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# NOTES ON MONADIC LOGIC. PART B: COMPLEXITY OF LINEAR ORDERS IN ZFC

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ABSTRACT

In those notes we prove in ZFC: (a) that the monadic theory of linear order (syntactically) interprets and has the same Lowenheim number as second order logic (the interpretation is semantical but not in the "classical" way), (b) a parallel (weaker) result for the monadic logic for completely metrizable spaces. The main results are in §§5, 6.

## §0. Introduction

For a survey and history see Gurevich [Gu].

We continue here [Sh42], [GuSh123], [GuSh143] and, in particular, [GuSh151], where we used weak instances of GCH (so that the proof does not work in ZFC) and quite saturated orders; topologically, those orders are very far from first countable spaces we use here. In [Sh205] we got the result for completely metrizable spaces — but again not in ZFC (essentially when $V = L$).

Note that in such interpretations we have two problems: to find models in which we can interpret much (see §2, §3), and to show that we can determine when the interpretation is essentially what we want, here mainly that a relevant order is a well ordering (see 4.4). Here our interpretations are not standard, so we interpret second order logic in a universe after appropriate forcing. But as the forcing adds no new short sequences of ordinals (i.e. the topology is $\kappa$-distributive for appropriate $\kappa$) we can go back to our original universe. The paper is self-contained.

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By [GuSh168] we cannot use classical interpretations for the real line. In general, we suggest using the following to get the same result for, e.g., the class of linear orders.

You work in a universe of set theory such that:

(*) for every regular $\lambda > \aleph_0$ and $A_i \subseteq \lambda$ for $i < \lambda$, there is a pressing down function $h$ such that:

(a) for $\alpha, i < \lambda$, if $A_i$ is stationary then so is $\{\zeta \in A_i : h(\zeta) = \alpha\}$;

(b) if $\delta < \lambda$, $\mathrm{cf}(\delta) > \aleph_0$ *then* there is a club $C_\delta$ of $\delta$ such that $(\forall \beta \in C_\delta)[h(\beta) = h(\delta)]$.

(This is quite easy to force.) Now combine [GuSh168] and [Sh42], §4.

**Problems on Monadic Logic**

**Problems of group $\alpha$**

(1) Is there a sentence in monadic logic, characterizing the real order up to isomorphism? Note, if this fails, then by Part A (i.e. [Sh284a]) the second order theory of the continuuum is necessarily the same in $V^P$ and $V^Q$ where

$$Q = \mathrm{Levy}(\aleph_0, \aleph_1),$$

$$P = \mathrm{Levy}(\aleph_0, \aleph_0) \quad \text{(i.e. Cohen forcing).}$$

(2) Is there a monadic formula $\varphi(X)$ such that for $X \subseteq \mathbf{R}$

$$(\mathbf{R}, <) \models \varphi[X] \quad \text{iff } X \text{ is countable}$$

(see [Gu1]). Now we know.

(3) (MA) Is the monadic theory of all $(A, <)$, ${}^{\omega>}2 \subseteq A \subseteq {}^{\omega\geqq}2$, such that for $\nu \in {}^{\omega>}2$, $|\{\eta : \nu < \eta \in A\}| = \aleph_1$ the same?

(4) What about the theory of topological spaces with a basis of clopen sets? (Under GCH, see [Sh205].)

(5) Show that the Borel monadic theory of the real line is decidable.

**Problems of group $\beta$**

(1) Show the consistency of: the monadic theory of well ordering is decidable and has Lowenheim number $\aleph_\omega$.

(2) Show the consistency of: the monadic theory of $\{({}^{\omega\geqq}\lambda, <) : \lambda\}$ has a small Lowenheim number.

(2)(A) Show that the monadic theory of $({}^{\omega\geqq}\lambda, <)$ is bi-interpretable with

$$\{\psi : \psi \text{ a second order sentence, } \Vdash_{\mathrm{Levy}(\aleph_0, \lambda)} \text{“}\aleph_0 \models \psi\text{”}\}.$$

(3) Similar questions on $({}^{\omega>}\lambda, <)$ in $L(Q^{\mathrm{pd}})$ (see [Sh205]).

## §1. Preliminaries

1.1. Definition. (0) $u$, $v$ vary over regular open non-empty sets of the relevant topology.

(1) For a topological space $X$ and a formula $\varphi(u, \dots)$, let

$$\mathrm{val}_u\, \varphi(u, \dots) = \bigcup\{u : \varphi(u, \dots) \text{ is satisfied}\}.$$

(2) A topological space $X$ is $\kappa$-weakly distributive *if* the union of $<\kappa$ nowhere dense subsets of $X$ is nowhere dense in $X$.

$X$ is $\kappa$-distributive *if* for every $\langle I_\alpha : \alpha < \alpha^* < \kappa\rangle$, where $I_\alpha$ is a maximal family of pairwise disjoint regular open non-empty subsets of $X$, there is an open $u \neq \emptyset$ such that $\bigwedge_\alpha (\exists\, u_\alpha \in I_\alpha)\, u \subseteq u_\alpha$.

(3) A topological space $Y$ has [*weak*] *distributivity* $\kappa$ *if* for every regular open $u$, $Y \restriction u$ is $\kappa$-[weak] distributive but not $\kappa^+$-[weak] distributive.

1.1A. Fact. A $\kappa$-distributive topological space is $\kappa$-weakly distributive. If the topology is induced by a dense linear order (on the points) *then* the inverse is true too.

1.2. Definition. For a topological space $X$, $M_X$ is the model with universe $\mathscr{P}(X)$ and relations $\subseteq$ (being a subset) and $\mathrm{Op} = \{u \subseteq X : u \text{ open}\}$. This we call the monadic topology (of $X$). We sometimes use $M_X$ instead of $X$ or $M = M_X$ instead of $X$.

1.3. Notation. Let PsOr (short for Pseudo Ordinals) be

$$\{(\alpha, q);\ \alpha \text{ an ordinal, } q \in \mathbf{Q}\ (\mathbf{Q} \text{ the rationals) such that:}$$
$$\text{if } \alpha \text{ is a limit ordinal of cofinality } \aleph_0 \text{ then } q \geqq 0\}$$

ordered lexicographically. We identify $(\alpha, 0)$ with $\alpha$. We use $\alpha, \beta$, etc. to denote members of PsOr. Let $(\alpha, q)^{[1]} = \alpha$ and $(\alpha, q)^{[2]} = q$. Let $T$ denote a set of sequences of members of PsOr, closed under initial segments. $T$ is a tree — by the order of being initial segments. For a sequence $\eta$ of length a successor ordinal let $\eta(\mathrm{lt}) \stackrel{\mathrm{def}}{=} \eta(\lg(\eta) - 1)$ [lt stands for “last”]. Let $\eta \trianglelefteq \nu$ mean $\eta$ is an initial segment of $\nu$, and $\eta \triangleleft \nu$ means $\eta \trianglelefteq \nu \ \&\ \eta \neq \nu$. Let

$$\mathrm{Rang}^{[l]}(\eta) = \{\eta(i)^{[l]} : i < \lg(\eta)\}.$$

1.4. DEFINITION. (1) For a tree $T$

(a) $< 15_{\mathrm{lx}}$ is the lexicographic order: $\eta \leqq_{\mathrm{lx}} \nu$ if $\eta < \nu$ or $\eta \restriction \alpha = \nu \restriction \alpha$, $\eta(\alpha) < \nu(\alpha)$ (where $\alpha < \lg(\eta)$, $\alpha < \lg(\nu)$, $\lg(\eta)$).

(b) (i) $\max(T) = \{\eta \in T : \text{for no } \nu \in T, \eta \lhd \nu\}$,

(ii) $\mathrm{nmax}(T) = T \setminus \max(T)$,

(iii) $\lim(T) = \{\eta \in T : \lg(\eta) \text{ is a limit ordinal}\}$,

(iv) $\mathrm{Mlim}(T) = \lim(T) \cap \max(T)$,

(v) $\mathrm{Clim}(T) = \{\eta \in \lim(T) : \lg(\eta) \text{ has cofinality } \aleph_0\}$.

(2) A tree $T$ is called *standard* if:

(a) for every $\eta \in T$, and $(\alpha, q_1) \in \mathrm{PsOr}$, $(\alpha, q_2) \in \mathrm{PsOr}$, we have: $\eta \ ^\wedge \langle(\alpha, q_1)\rangle \in T \Leftrightarrow \eta \ ^\wedge \langle(\alpha, q_2)\rangle \in T$,

(b) if $\eta \ ^\wedge \langle \alpha \rangle \in T$ and $\beta < \alpha$, then $\eta ^\wedge \langle \beta \rangle \in T$.

1.5. DEFINITION. Let Y be a topological space, $D \subseteq Y$, $P \subseteq Y$, and $E_1$, $E_2 \subseteq D$.

(1) We say $P$ is $(D, E_1, E_2)$*-perfect if*: $P$ is closed, has no isolated point (in the induced topology), $P \cap D \subseteq E_1 \cup E_2$, and $P \cap E_1$, $P \cap E_2$ are dense in $P$.

(2) We say $P$ is a strongly $(D, E_1, E_2)$-perfect set *if* it is $(D, E_1, E_2)$-perfect and $P \setminus D$ is dense in $P$.

(3) We say $P$ is a hereditary strongly $(D, E_1, E_2)$-perfect set *if* it is $(D, E_1, E_2)$-perfect but for every $(D, E_1, E_2)$-perfect $P' \subseteq P$ we have $P' \setminus D \neq \varnothing$.

1.6. DEFINITION. In a topological space $Y$, for subsets $X_1$, $X_2$ we let:

(i) $X_1 \equiv X_2$ iff $(X_1 - X_2) \cup (X_2 - X_1)$ is nowhere dense,

(ii) $X_1 \subseteq^* X_2$ iff $X_1 - X_2$ is nowhere dense.

## §2. Quite distributive linear order for which wonder sets exist

2.1. DEFINITION. For $T$ (as in 1.3, of course), $\mathrm{Top}_{\mathrm{lx}}(T)$ is the topology induced on $T$ by the linear order $<_{\mathrm{lx}}$ (i.e. the topology with the set of open intervals as a basis).

In this section we use only the topology from 2.1.

We now define the topologies we shall mainly use (main case: $\zeta = \kappa$).

2.2. DEFINITION. For cardinal $\lambda$, ordinal $\zeta < \lambda$ and non-empty sets of limit ordinals $S_1 \subseteq \lambda$, $S_2 \subseteq \lambda$, letting $\bar{p} = \langle \lambda, \zeta, S_1, S_2 \rangle$ we define $T$, $D_i$ $(i \in S_2)$, $D$, $D_a$ $(a \subseteq S_2)$, $Y$ (more exactly $T = T(\bar{p})$, etc.) by

$T = \{\eta : \eta$ is a sequence of elements $x \in \mathrm{PsOr}$, where $x^{[1]}$ *is smaller than* $\lambda + 1$, $\eta$ has length $< \zeta$ and is such that:

(i) for no limit ordinals $\delta < \lg(\eta)$, $\sup\{\eta(i)^{[1]} + 1 : i < \delta, \eta^{(i)^{[1]}} < \lambda\} \in S_1$,

(ii) for no $\alpha + 1 < \lg(\eta)$, $\eta(\alpha)^{[1]}$ is in $S_2$,

(iii) if $\delta < \lg(\eta)$, $\mathrm{cf}(\delta) = \aleph_0$ then
$\eta(\delta)^{[1]} = 0 \Rightarrow \eta(\delta)^{[2]} > 0$,
$\eta(\delta)^{[1]} = \lambda \Rightarrow \eta(\delta)^{[2]} \leqq 0$,

(iv) if $\delta + 1 < \lg(\eta)$, $\mathrm{cf}(\delta) = \aleph_0$ then
$\eta(\delta)^{[1]} = \lambda \Rightarrow \eta(\delta)^{[2]} < 0$,

(v) if $\eta(\alpha)^{[1]} \in S_2$ then $\eta(\alpha)^{[2]} = 0$.

$D_i \stackrel{\text{def}}{=} \{\eta \in T : i = \eta(\mathrm{lt})^{[1]}\}$ for $i \in S_2$ (so $\eta \in D_i \Rightarrow \lg(\eta)$ is a successor ordinal),

$D \stackrel{\text{def}}{=} \bigcup_{i \in S_2} D_i$, for $a \subseteq S_2$, $D_a \stackrel{\text{def}}{=} \bigcup_{i \in a} D_i$ (no confusion will arise with $D_i$),

$Y \stackrel{\text{def}}{=} \max(T) \cup \lim_2(T)$ where $\lim_2(T) = \{\eta : \lg(\eta)$ has the form $\delta$, $\mathrm{cf}\,\delta = \aleph_0$, $\eta \notin \mathrm{Mlim}(T)\}$, we identify it with the subspace induced by $\mathrm{Top}_{\mathrm{lx}}(T)$ on $Y$.

For $\eta \in T$ let $\zeta(\eta) = \sup\{\eta(i)^{[1]} + 1 : i < \lg(\eta), \eta^{[1]}(i) < \lambda\}$.

2.2A. Remark. (1) Note that:

$$\max(T) = \mathrm{Mlim}_1(T) \cup \mathrm{M}_2(T) \cup D \qquad \text{(disjoint union)}$$

where

$$\mathrm{Mlim}_1(T) = \{\eta \in T : \delta = \lg(\eta) \text{ is limit and } \sup\{\eta(i)^{[1]} + 1 : i < \delta, \eta(i)^{[1]} < \lambda\} \in S_1\},$$

$$\mathrm{M}_2(T) = \{\eta \in T : \lg(\eta) \text{ has the form } \delta + 1, \mathrm{cf}(\delta) = \aleph_0 \text{ and } \eta(\delta) \text{ is } (\lambda, 1)\}.$$

(2) We could have added in the definition of $T$:
(v) $\lg(\eta) = \delta + 1$, $\delta \in S_2 \Rightarrow \eta(\delta) = (0, 0)$.

2.3. Fact. (1) $T(\bar{p})$ is (by $<_{\mathrm{lx}}$) dense in itself (here we use the density of $\mathbf{Q}$),

(2) if $\zeta$ is limit or the successor of a limit ordinal *then* each $D_i$ is a dense subset of $T(\bar{p})$ (hence $D$ and $Y$ are),

(3) if $(\forall \delta \in S_1 \cup S_2)$ $\mathrm{cf}\,\delta = \aleph_0$, *then* $Y$ satisfies first countability axiom (here we use $\mathbf{Q}$ and the case "$\mathrm{cf}\,\delta = \aleph_0$" in the definition of PsOr and (iii) and (iv) in the Definition of $T$ in 2.2),

(4) $Y$ is dense in itself and Hausdorff,

(5) in $Y$ "almost" every monotonic $\omega$-sequence $\langle \eta_n : n < \omega \rangle$ has a limit — the exception satisfies for some $\nu$ and $\alpha$, for some $n_0$ for $n \geqq n_0$, $\nu < \eta_n$, $\eta_n(\lg(\nu)) = (\alpha, q_n)$ and $\langle q_n : n_0 \leqq n < \omega \rangle$ is monotonic; similarly in $T$,

(6) if $P \subseteq Y$ is closed dense in itself, $E_n \subseteq P$ dense in $P$ for $n < \omega$, $E_0 \subseteq Y \setminus D$ then there is $P' \subseteq P$ closed dense in itself, $E_n \cap P'$ dense in $P$ and $P' \cap D$ is countable and for some $\delta < \lambda$,
$\mathrm{cf}\,\delta = \aleph_0\ [\eta \in P' \setminus D \Rightarrow \lg(\eta) = \delta]$,
$[\eta \in P' \setminus D \Rightarrow \lg(\eta) = \delta + 1\ \&\ \eta(\delta) = (\lambda, 0)]$,
$[\eta \in P' \cap D \Rightarrow \lg(\eta) < \delta]$.

(7) The $P'$ in (6) satisfies: for every perfect $P'' \subseteq P'$, $P'' \setminus D$ is dense in $P''$.

2.4. Claim. (1) Suppose $\lambda > \kappa^+$ and $\lambda$ and $\kappa$ are regular cardinals. Then there is $S_1 \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \aleph_0\}$ such that:

(*) the set $\{\delta < \lambda : \mathrm{cf}(\delta) = \kappa,\ S_1 \cap \delta$ is not a stationary subset of $\delta\}$ is stationary
(if $\kappa = \aleph_0$, this says nothing).

(2) If $\lambda$, $\kappa$, $S_1$ are as in (1), $S_2 \subseteq \lambda$ is a set of limit ordinals and $(\forall \alpha < \lambda)[|\alpha|^\kappa < \lambda]$ *then* the distributivity of $T = T[(\lambda, \kappa, S_1, S_2)]$ and of $Y$ is exactly $\kappa$.

(3) Suppose $\lambda = \mathrm{cf}(\lambda) > \zeta$, $\zeta \in \{\xi, \xi + 1\}$, $\xi$ limit, $\kappa \geqq \mathrm{cf}(\xi)$, $S_1$ and $S_2$ are sets of limit ordinals $< \lambda$, the set $\{\delta < \lambda : \mathrm{cf}\,\delta = \kappa, S_1 \cap \delta$ not stationary (in $\delta$)$\}$ is stationary, and $T = T[(\lambda, \zeta, S_1, S_2)]$. If $\forall \alpha < \lambda[|\alpha|^\kappa < \lambda]$ *then* in the following game player I has no winning strategy: a play lasts $\mathrm{cf}(\xi)$ moves, in the $i$th move player I chooses an open $u_{2i}$ (in the topological space $\mathrm{Top}_{\mathrm{lx}}(T)$), $u_{2i} \subseteq \bigcap_{j<2i} u_j$, $u_{2i} \cap \mathrm{Mlim}(T) \neq \emptyset$, and player II chooses open $u_{2i+1} \subseteq u_{2i}$ such that $u_{2i+1} \cap \mathrm{Mlim}(T) \neq \emptyset$. Player I wins if for some $i < \mathrm{cf}(\xi)$ he has no legal move.

Proof. (1) Look at [Sh237e] Lemma 4 (p. 278); we can rephrase it as follows.

2.4A. Lemma. *Let $\lambda > \aleph_0$ be regular, $R$ be a set of regular cardinals, $(\forall \kappa \in R)\kappa^+ < \lambda$, and*

$$\langle S^*_\kappa : \kappa \in R \rangle \text{ be such that } S^*_\kappa \subseteq \{\delta < \lambda : \mathrm{cf}\,\delta = \kappa\} \text{ stationary.}$$

*Then we have $S_\kappa$ $(\kappa \in R)$ such that:*

(a)′ *$S_\kappa \subseteq S^*_\kappa$ is stationary (as a subset of $\lambda$),*

(c)′ *if $\delta \in S_\kappa$, $\kappa \in R$ then $\delta \cap (\bigcup\{S_\mu : \mu \in R \cap \kappa\})$ is not a stationary subset of $\delta$.*

[The changes in the proof are minor. Choose $S(\kappa, i) \subseteq S^*_\kappa$, and define $T_\xi$ in (iv) (in [Sh237e], Lemma 4) as

$$T_\xi = \{\delta : \delta \in \bigcup\{S(\kappa_\xi, i) : i \notin \langle \gamma^\zeta_{\kappa_\xi} : <\xi\}\} \text{ and}$$

$$\bigcup\{S^\xi_\kappa : \kappa \in R \cap \kappa_\xi\} \text{ is not stationary in } \delta\}$$

(and in $279^{7\text{-}9}$ change $\kappa_a^+$, $\kappa_\alpha$, $\kappa$, $\kappa_0^+$, $\kappa^+$ to $\kappa_\alpha^+$, $\kappa_\alpha$, $\kappa_\alpha^+$.]

Continuation of the Proof of 2.4. (2) Follows by 2.4(3) (which is stronger — it suffices that player I does not win any such game of length $\alpha < \kappa$).

(3) Left as an exercise.

2.4B. Remark. (1) In 2.4(2) instead of ($\lambda$ is regular and),$(\forall\alpha < \lambda)|\alpha|^\kappa < \lambda$, it suffices to assume ($\lambda$ regular and):

(a) $\lambda^\kappa = \lambda$ or even,

(b) there is a stationary $S^* \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$ which is in $I[\lambda]$ (i.e. good, see [Sh108], or better, [Sh88], Appendix, and then use $S_1 \subseteq S^*$).

(2) In 2.4(3) instead of $(\forall\alpha < \lambda)[|\alpha|^\kappa < \lambda]$ it is enough to assume $\{\delta < \lambda : S_1 \cap \delta$ is not stationary, $\mathrm{cf}\,\delta = \kappa\}$ contains a stationary good set.

(3) Remember that if $\lambda = \mu^+$, $\mu$ regular, then $\{\delta < \lambda : \mathrm{cf}\,\delta < \mu\} \in I[\lambda]$ (see [Sh 300a] or [Sh 351, 4.1]).

2.5. Main Construction Lemma. *Suppose*

(*) $\bar{p} = (\lambda^+, \kappa, S_1, S_2)$, $\kappa > \aleph_0$ *is regular*, $\lambda = \lambda^\kappa$, $S_1 \subseteq \{\delta < \lambda^+ : \mathrm{cf}(\delta) = \aleph_0, \delta > \kappa\}$ *is stationary*, $S_2 = \{i + \omega : i < \lambda\}$.

Then *for every equivalence relation* $\mathscr{E}$ *on* $S_2$, *there are* $W, W^+ \subseteq \mathrm{Mlim}(T)$ *such that for any* $E \subseteq D$ *and open* $\mathscr{U}_0 \subseteq Y$ *the following are equivalent*:

(a) *if* $\mathscr{U}_1$ *is an open subset of* $\mathscr{U}_0$ *and* $E_1, E_2 \subseteq E$ *are dense in* $Y[\bar{p}] \cap \mathscr{U}_1$ then:

(a1) *for some strong* $(D, E_1, E_2)$*-perfect* $P$, $P \setminus D \subseteq W \cap \mathscr{U}_1$ *but*

(a2) *for no strong* $(D, E, E)$*-perfect* $P$, $P \setminus D \subseteq W^+$;

(b) $\mathrm{val}_{\mathscr{U}}[\bigvee_{i \in S_2} E \cap \mathscr{U} \subseteq D_{i/\mathscr{E}} \cap \mathscr{U}]$ *is dense in* $Y \cap \mathscr{U}_0$;

(c) *like* (a) *but we replace* (a1) *by the negation of*:

(a1)′ *for every strongly* $(D, E_1, E_2)$*-perfect* $P$, $P \subseteq \mathscr{U}_1$ *there is* $P_1 \subseteq P$ *which is strongly* $(D, E_1, E_2)$*-perfect and* $P_1 \setminus D$ *is disjoint from* $W$ (*but not empty*).

2.5A. Remark. (1) If $\mathscr{E}$ has $< \kappa$ equivalence classes then we can omit (a2) while retaining the equivalence.

(2) We could, of course, restrict ourselves to $E$ dense in $\mathscr{U}_0$.

2.6. LEMMA. *Let* $\lambda > \kappa + \aleph_0$, $\kappa = \mathrm{cf}(\kappa) \geqq \aleph_0$, $S_1 \subseteq \{\delta < \lambda^+ : \mathrm{cf}(\delta) = \aleph_0\}$, $\lambda = \lambda^\kappa$, $S_1$ *stationary*, $S_2 = \{i + \omega : i < \lambda\}$.

*Then the conclusion of* 2.5 *holds also for* $T = T(\bar{p})$ *when* $\bar{p} = (\lambda^+, \kappa + 1, S_1, S_2)$.

REMARK. The main addition is $\kappa = \aleph_0$.

PROOF. Like the proof of 2.5.

PROOF OF 2.5. Let $\{\langle N_l^\alpha : l \leqq \omega\rangle : \alpha < \alpha^*\}$ and the functions $\zeta$, $h_{\alpha,\beta}$ be from the black box for $\lambda, \kappa$, the stationary set $S_1 \subseteq \lambda^+$ and $A = \lambda^+ \cup T$ (see 1 of the Appendix), so $h_{\alpha,\beta}$ is the isomorphism from $N_\omega^\alpha$ onto $N_\omega^\beta$ when $\zeta(\alpha) = \zeta(\beta)$. Note that $h_{\alpha,\beta}$ is the identity on $N_\omega^\alpha \cap N_\omega^\beta \cap \lambda$. For every $\alpha$ we define $P_\alpha$, perfect or empty. The definition is split into three cases.

We let $N^\alpha \overset{\mathrm{def}}{=} N_\omega^\alpha$.

*Case A*. There are $\beta$, $i$, $E_1$, $E_2$ $\mathscr{u}$ and $a$ such that:

(i) $\zeta(\beta) = \zeta(\alpha)$ and $i = i_{\zeta(\alpha)} \in S_2$, the sets $E_1, E_2 \subseteq D_{i/\mathscr{E}}$ are dense in $Y$ ($\mathscr{E}$ is the equivalence relation on $S_2$), $\mathscr{u}$ is an open set of $Y$, and $a \subseteq S_2$, $|a| \leqq \kappa$,

(ii) we have

$$N_\omega^\beta < M \overset{\mathrm{def}}{=} \Big(\lambda^+ \cup T, \lambda, i, \vartriangleleft, <, <_{\mathrm{lx}}, E_1, E_2, D_i,$$
$$\mathscr{E}, \{(\alpha, \eta, \eta(\alpha)) : \eta \in T, \alpha < \lg(\eta)\},$$
$$\{\langle j, x\rangle : x \in D_j\}, a, \gamma, Y, \mathscr{u}, \bigcup_{j \in a} D_j\Big)_{\gamma \in a \cup (\kappa + 1)},$$

(iii) $[\eta \in T \cap N_\omega^\beta \Rightarrow \{\eta \restriction i : \leqq \lg(\eta)\} \subseteq N_\omega^\beta]$.

We choose the minimal such $\beta$, and any such $M$ (but such that $M$ etc. depend on $\zeta(\alpha)$ only, rather than on $\alpha$!). Let $\gamma_n \in \lambda^+ \cap N^\beta$, $\lambda < \gamma_n < \gamma_{n+1}$, $\bigcup_{n<\omega} \gamma_n = \sup(N^\beta \cap \lambda^+)$. We now define by induction on $n$, for every $\rho \in {}^n\omega$, a sequence $\eta_\rho$ and ordinal $j_\rho$ such that:

(i) $\eta_\rho \in T \cap N^\beta$ and $\eta_\rho \wedge \langle j_\rho + \omega\rangle \in T \cap N^\beta$ and $j_\rho$ is a successor ordinal,

(ii) $\eta_\rho \wedge \langle j_\rho + \omega\rangle$ is in $E_1$ if $l$ is even and in $E_2$ if $l$ is odd,

(iii) *if* $n = m + 1$, *then* $\eta_{\rho \restriction m} \wedge \langle j_\rho + \rho(m)\rangle$ is an initial segment of $\eta_\rho$,

(iv) $\sup(\mathrm{Rang}^{[1]}(\eta_\rho)) \geqq \gamma_{n-1}$ when $n > 0$,

(v) $\{\eta \in Y : \eta_{\langle\,\rangle} \vartriangleleft \eta\} \subseteq \mathscr{u}$.

There is no problem to do this (remembering that $N^\beta < M$). Let $\eta_\rho^\alpha = h_{\beta,\alpha}(\eta_\rho)$ for $\rho \in {}^{\omega>}\omega$ (so $\eta_\rho^\beta = \eta_\rho$, and if $\zeta(\alpha) = \zeta(\gamma) = \zeta(\xi)$ then $h_{\gamma,\xi}(\eta_\rho^\gamma) = \eta_\rho^\xi$). Let for

$\rho \in {}^{\omega}\omega$ and $\alpha$, $\eta^{\alpha}_{\rho} \overset{\text{def}}{=} \bigcup_{m<\omega} \eta^{\alpha}_{\rho \restriction m}$ — so it has limit length $\zeta(\alpha)$ $(\in S_1)$ so $h^{a}_{r} \in$ Mlim $T(\bar{p}) \subseteq Y$, and:

$$P_{\alpha} = \{\eta^{\alpha}_{\rho} \,{}^{\wedge}\, \langle j_{\rho} + \omega \rangle : \rho \in {}^{\omega >}\omega\} \cup \{\eta^{\alpha}_{\rho} : \rho \in {}^{\omega}\omega\}.$$

Check that $P_{\alpha}$ is as required.

*Case B.* There are $\beta$, $i$, $E$, $\mathscr{u}$ and $a$ such that:

(i) $\zeta(\beta) = \zeta(\alpha)$ and $E \subseteq D$ is dense in $D$, $\mathscr{u}$ an open subset of $Y$ and $a \subseteq S_2$, $|a| \leqq \kappa$,

(ii) for no $\mathscr{u}' \subseteq \mathscr{u}$, $n < \omega$, $i_1, \ldots, i_n \in S_2$ is $E \cap \mathscr{u}'$ included in $D_{i_1/\mathscr{E}} \cup \cdots \cup D_{i_n/\mathscr{E}}$,

(iii) $N^{\beta}_{\omega} < M \overset{\text{def}}{=} (\lambda^{+} \cup T, \lambda^{+}, <, \lhd, <_{\mathrm{lx}}, E, \{(x, j) : x \in D_j\},$
$\mathscr{E}, \{\langle \alpha, \eta, \eta(\alpha) \rangle : \eta \in T, \alpha < \lg\alpha\}\ a, \varepsilon, Y, \bigcup_{j \in a} D_j)_{\varepsilon \in a \cup (\kappa+1)}$,

so

(iv) $\kappa + 1 \subseteq N^{\beta}_{0}$, hence for $n \leqq \omega$

$$[\eta \in T \cap N^{\beta}_{n} \Rightarrow \{\eta \restriction i : i \leqq \lg(\eta)\} \subseteq N^{\beta}_{n}].$$

[Note: As $N^{\beta}_{\omega}$, $M$ have the same vocabulary, Cases A, B are disjoint.]

We choose the minimal such $\beta$ (depending on $\zeta(\alpha)$ only) and any such $M$. Let $\gamma_n \in \lambda^{+} \cap N^{\beta}$, $\lambda < \gamma_n < \gamma_{n+1}$, $\bigcup_{n<\omega} \gamma_n = \sup(N^{\beta} \cap \lambda^{+})$. We now define by induction on $n$ for every $\rho \in {}^{n \geqq}n$ a sequence $\eta_{\rho}$ and ordinals $j_{\rho}$, $i_{\rho}$ such that:

(i) $\eta_{\rho} \in T \cap N^{\beta}$, $\eta_{\rho} \,{}^{\wedge}\, \langle j_{\rho} + \omega \rangle \in T \cap N^{\beta}$, $j_{\rho}$ is a successor ordinal,

(ii) $\eta_{\rho} \,{}^{\wedge}\, \langle j_{\rho} + \omega \rangle \in D_{i_{\rho}} \cap E$,

(iii) $\rho \neq \nu \Rightarrow i_{\rho}/\mathscr{E} \neq i_{\nu}/\mathscr{E} \wedge \eta_{\rho} \not\trianglelefteq \eta_{\nu}$,

(iv) if $m < \lg(\rho)$ then $\eta_{\rho \restriction m} \,{}^{\wedge}\, \langle j_{\rho} + \rho(m) \rangle$ is an initial segment of $\eta_{\rho}$,

(v) $\sup(\mathrm{Rang}^{[1]}(\eta_{\rho})) \geqq \gamma_{\lg(\rho)}$ when $\lg(\rho) > 0$,

(vi) $\{\eta \in Y : \eta_{\langle\ \rangle} \lhd \eta\} \subseteq \mathscr{u}$.

We continue as in Case A.

*Case C.* Neither Case A nor Case B.

Let $P_{\alpha} = \varnothing$.

So the $P_{\alpha}$'s are defined.

Let $t_{\alpha} = \{\eta \restriction \gamma : \eta \in P_{\alpha} \cap \text{Mlim } T, \gamma < \lg(\eta)\}$; it is a tree, and if $\zeta(\alpha) = \zeta \Rightarrow P_{\alpha} \neq \varnothing$ let $s_{\zeta} = \bigcup\{t_{\alpha} : \zeta(\alpha) = \zeta\}$. Now each $t_{\alpha}$ is a tree, and [by (B)(c) of Theorem 1 of the Appendix] also $s_{\zeta}$ is a tree. Also, by the same clause, if $\eta \in t_{\alpha} \setminus t_{\beta}$, $\nu \in t_{\beta} \setminus t_{\alpha}$, $\zeta(\alpha) = \zeta(\beta)$, $\eta(\xi) \neq \nu(\xi)$, $\eta \restriction \xi = \nu \restriction \xi$, then $\eta \restriction \xi$ is not a splitting point of $t_{\alpha}$ (i.e. does not belong to $\{\eta^{\alpha}_{\rho} : \rho \in {}^{\omega >}\omega\}$); it thus holds because $j_{\rho} \in S_2 \subseteq \lambda$. Note (we use the last sentence for $\bigoplus$(b) below):

$\bigoplus$(a) if $\eta \in P_\alpha$ then sup rang$(\eta) \leqq \zeta(\alpha)$, and equality holds when $\eta \notin D$.

(b) if $\zeta \in S_1$, $\eta \neq \nu \in \bigcup\{P_\alpha : \zeta(\alpha) = \zeta\}$, $\eta \restriction \xi = \nu \restriction \xi$, $\eta(\xi) \neq \nu(\xi)$ *then*: $\eta(\xi)$, $\nu(\xi) \geqq \lambda$ or

$\eta(\xi) + \omega = \nu(\xi) + \omega \in S_2 \subseteq \lambda$,

$(\eta \restriction \xi) \wedge \langle \eta(\xi) + \omega \rangle \in D_i$ where $i = i_{\zeta(\alpha)}$ in Case A, $i = i_{\eta \restriction i}$ in Case B.

Now let $W^+ = \bigcup\{P_\alpha : \alpha < \alpha(*)$, for $\alpha$ Case B occurs$\} \setminus D$.

Note:

$\bigoplus_0$ if for $\alpha$ Case A or B occurs then $P_\alpha$ is strongly $(D, E_1^{N_\alpha}, E_2^{N_\alpha})$-perfect,

$\bigoplus_1$ for open $\mathscr{u} \subseteq Y$ and $E \subseteq D$, $E \cap \mathscr{u}$ is dense in $\mathscr{u}$, the following are equivalent:

(α) for every $\mathscr{u}' \subseteq \mathscr{u}$, letting $E' = E \cap \mathscr{u}'$, there is a $(D, E', E')$-perfect $P, P \setminus D \subseteq W^+$,

(β) for no $\mathscr{u}' \subseteq \mathscr{u}$, $n < \omega$, $i_1, \ldots, i_n \in S_2$ is $E \cap \mathscr{u}' \subseteq D_{i_1/\mathscr{E}} \cup D_{i_2/\mathscr{E}} \cup \cdots \cup D_{i_n/\mathscr{E}}$.

We leave that to the reader and a similar argument is advanced below [(β)$\Rightarrow$(α) by (C) of Theorem 1 of the Appendix and our choice of $P_\alpha$ in Case B; $\neg$ (β)$\Rightarrow$ $\neg$(α) as in the proof of "why is (*) enough"].

Let $W = \bigcup\{P_\alpha : \alpha < \alpha(*)$, for $\alpha$ Case A occurs$\} \setminus$ D. Now in the lemma, (b)$\Rightarrow$(a) was taken care of (by the choice of the $N^\alpha$'s (i.e. part (C) of Theorem 1 of the Appendix) and the $P_\alpha$'s and $\bigoplus_1$). Now (a)$\Rightarrow$(c) is trivial. So assume (b) fail for the pair $E$, $\mathscr{u}_0$ and we shall prove that (c) fails. For this it suffices to assume that (a2) holds and show that (a1)′ fails. So there is an open subset $\mathscr{u}$ of $Y \cap \mathscr{u}_0$, $\mathscr{u} \neq \varnothing$, and for no open non-empty $\mathscr{u}' \subseteq \mathscr{u}$, $(\exists i)[E \cap \mathscr{u}' \subseteq D_{i/\mathscr{E}}]$.

(*) there is a non-empty open $\mathscr{u}_1 \subseteq \mathscr{u}$ and dense disjoint $E_1, E_2 \subseteq E \cap \mathscr{u}_1$ such that for no $i \in S_2$,

$E_1 \cap D_{i/\mathscr{E}} \neq \varnothing \wedge E_2 \cap D_{i/\mathscr{E}} \neq \varnothing$.

*Why is* (*) *enough*?

We shall show that $E_1$, $E_2$, $\mathscr{u}_1$ exemplify the failure of (c) (as (c) for $E$, $\mathscr{u}_0$ implies its version for $E$, $\mathscr{u}_1$). I.e. we prove that (a1)′ holds for $E_1$, $E_2$, $\mathscr{u}_2$. Suppose $P$ is a strongly $(D, E_1, E_2)$-perfect set, $P \setminus D \subseteq W \cap \mathscr{u}_1$ or just contradicting (a1)′. Let $\zeta(P) = \mathrm{Min}\{\zeta : P \setminus D \subseteq \bigcup_{\zeta(\alpha) \leqq \zeta} P_\alpha\}$ and choose $P$ with minimal $\zeta(P)$ (which is a strongly $(D, E_1, E_2)$-perfect set, contradicting (a1)′). W.l.o.g. by 2.3(6) $P \cap D$ is a countable dense subset of $P$, hence also $P \setminus D$ has a countable dense subset. Trivially $\zeta$ is a limit ordinal [each $\zeta(\alpha)$ is a limit ordinal]. Also its cofinality is $\aleph_0$. [Otherwise, as $\bigwedge_\alpha \zeta(P_\alpha) \neq \zeta$ and $P \setminus D$ has a countable dense subset, for some $\zeta(*) < \zeta$, $(P \setminus D) \cap \bigcup_{\zeta(\alpha) \leqq \zeta(*)} P_\alpha$ is dense in $P \setminus D$. Hence by $\bigoplus$(a) for a dense subset of $\eta \in P \setminus D$ we have

$\sup(\lambda \cap \mathrm{Rang}^{[1]}(\eta)) \leqq \zeta(*)$, hence for every $\eta \in P \setminus D$, we have $\sup(\lambda \cap \mathrm{Rang}^{[1]}(\eta)) \leqq \zeta(*)$; as $\kappa \leqq \zeta(*)$ (by $\oplus$(a) $\zeta(*)$ is in the closure of the range of the function $\zeta$ which is a subset of $S_1$ and in 2.5 we assume $S_1 \cap \kappa = \varnothing$). Also for every $\eta \in P$, we have $\sup(\lambda \cap \mathrm{Rang}^{[1]}(\eta)) \leqq \zeta(*)$. However, again by $\oplus$(a) this implies $P \setminus D \subseteq \bigcup_{\zeta(\alpha) \leq \zeta(*)} P_\alpha$, contradicting $\zeta(*) < \zeta$ and the minimality of $\zeta$]. W.l.o.g. $P \setminus D \subseteq \bigcup\{P_\alpha : \zeta(\alpha) = \zeta\}$. So Case A holds for $\alpha$ when $\zeta(\alpha) = \zeta$ and let $i(\zeta)$ be the $i$ which appears there (it does not depend on $\alpha$). W.l.o.g. $D_{i(\zeta)/\mathscr{E}} \cap E_1 = \varnothing$: otherwise exchange $E_1$, $E_2$ (remember we are assuming (*)).

Let $\zeta = \bigcup_{n<\omega} \gamma_n$, $\gamma_n < \gamma_{n+1}$.

We define by induction on $n < \omega$, $\eta_\rho, j_\rho$ for $\rho \in {}^n\omega$ such that:

(i) $\eta_\rho \,\hat{}\, \langle j_\rho + \omega \rangle \in E_1 \cap P$,

(ii) if $n = m + 1$, $j_\rho + \rho(m) < \eta_\rho(\lg(\eta_{\rho \restriction m})) < j_\rho + \omega$,

(iii) $\lg(\eta_\rho) \geqq \gamma_n$.

There is no problem [remembering that $P \cap E_1$ is dense in $P$, and by the choice of $\zeta$, for each $n < \omega$, $A_n = \{\eta \in P \setminus D : \sup \mathrm{Rang}(\eta) > \gamma_n\}$ is dense in $P$, and if $\eta \in A_n$, $\beta < \lg(\eta)$ then there is $\nu \in P \cap E_1$, $\eta \restriction \beta \lhd \nu$ and $E_1 \subseteq D$.

Let, for $\rho \in {}^\omega\omega$, $\eta_\rho = \bigcup \eta_{\rho \restriction n}$, so $\eta_\rho \in P$, $\sup \mathrm{Rang}(\eta_\rho) = \zeta$, hence $\eta_\rho \in P \setminus D$, so $\eta_\rho \in \bigcup\{P_\alpha : \zeta(\alpha) = \zeta\}$. Let $\rho_1 \neq \rho_2 \in {}^\omega\omega$; assume $\eta_{\rho_1}$ and $\eta_{\rho_2}$ belongs to $W$; look when $\eta_{\rho_1}, \eta_{\rho_2}$ split and get a contradiction to $\oplus$(b). In fact we get $\{\eta_\rho : \rho \in {}^\omega\omega\} \cap [\bigcup\{P_\alpha : \zeta(\alpha) = \zeta\}]$ has at most one element; we can get rid of it easily by replacing $P$ by some $(D, E_1, E_2)$-perfect set $P' \subseteq P$.

So (*) suffices.

*Why is* (*) *true*?

Suppose first for some $\mathscr{u}_1 \subseteq \mathscr{u}$, $n < \omega$, $i_1, \ldots, i_n \in S_2$, $E \cap \mathscr{u}_1 \subseteq \bigcup_{l=1}^n D_{i_l/\mathscr{E}}$, then (by shrinking $\mathscr{u}_1$ further), w.l.o.g. for $l = 1, \ldots, n$, $D_{i_l/\mathscr{E}} \cap E \cap \mathscr{u}_1$ is dense in $\mathscr{u}_1$. If $n = 1$ we contradict the assumption "not (b)" ($n = 0$ — impossible); if $n \geqq 2$ let, for $l = 1, 2$, $E_l = E \cap \mathscr{u}_1 \cap D_{i_l/\mathscr{E}}$; they are as required. So suppose there are no such $\mathscr{u}_1, n, i_l \in S_2$ ($l = 1, n$). By $\oplus_1$ we can show (a2) fails, hence (c) fails.

2.7. Claim. In 2.5 we also get:

For every $S \subseteq S_2$, $S = \bigcup_{i \in S} i/\mathscr{E}$ *if* $E_1 \subseteq D_S$, $E_2 \subseteq D_{S_2 \setminus S}$, $P$ is $(D, E_1, E_2)$- perfect, *then* for some $(D, E_1, E_2)$-perfect $P_1 \subseteq P$, $P_1$ is disjoint from $W$.

Proof. By the proof of "Why is (*) enough" above.

## §3. Interpretability in the special topologies

3.1. LEMMA. *For any vocabulary* $L = \{R_l, F_m : l < n_p^L, m < n_f^L\}$ ($R_l$ *is an* $n(R_l)$*-place predicate symbol*, $F_m$ *an* $n(F_m)$*-place function symbol*), *there are monadic formulas*

$$\psi_{R_l}^L(\mathscr{u}, X_1, \ldots, X_{n(R_l)}, \bar{W}, D, D^*),$$

$$\psi_{F_m}^L(\mathscr{u}, X_1, \ldots, X_{n(F_n)+1}, \bar{W}, D, D^*)$$

*such that*:

(∗) *if* $T$, $D$, $D_i$ $(i \in S_2)$, $Y = \max(T) \cup \lim_2(T) \subseteq D^* \subseteq T$ *satisfies the conclusion of Main Lemma* 2.5, $M = M_{\mathrm{TOP}_{\mathrm{lt}}(T)} \upharpoonright D^*$, *S a subset of* $S_2$ *and N is an L-model with universe S*, then *for some sequence* $\bar{W}^N$ *of subsets of Y of length* $\lg(\bar{W})$:

(a) *for every* $l < n_p^L$ *and* $X_1, \ldots, X_{n(R_l)} \subseteq D^*$:
$M \models \psi_{R_l}^L(\mathscr{u}, X_1, \ldots, X_{n(R_l)}, \bar{W}^N, D)$ iff

$$\mathscr{u} \subseteq^* \mathrm{val}_{\mathscr{o}}[\vee\{\bigwedge_{k=1}^{n(R_l)} X_k \cap \mathscr{o} = D_{\alpha_k} \cap \mathscr{o} : \alpha_1, \ldots, \alpha_{n(R_l)} \in S \text{ and } N \models R_l[\alpha_1, \ldots, \alpha_{n(R_l)}]\}],$$

(b) *for every* $m < n_f^L$ *and* $X_1, \ldots, X_{n(F_m)} \subseteq D^*$:
$M \models \psi_{F_m}^L(\mathscr{u}, X_1, \ldots, X_{n(F_m)}, \bar{W}^N, D)$ iff

$$\mathscr{u} \subseteq^* \mathrm{val}_{\mathscr{o}}[\vee\{\bigwedge_{k=1}^{n(F_m)+1} X_k \cap \mathscr{o} = D_{\alpha_k} \cap \mathscr{o} : \alpha_1, \ldots, \alpha_{n(F_m)+1} \in S, \text{ and } N \models F_m[\alpha_1, \ldots, \alpha_{n(F_m)}] = \alpha_{n(F_m)+1}\}].$$

3.1A. NOTATION. (1) The relativization of $\psi_{R_l}^L$, $\psi_{F_m}^L$ to a predicate $D^*$ is denoted similarly with the added $D^*$ at the end. We shall use only those variants.

(2) We can replace $S$ by any subset of the same cardinality.

PROOF. Straightforward by 2.5, like [Sh42], §7[†] (or see [Gu] or [GuSh151] or [Sh284a], §1, §2).

## §4. The interpretation and recovering the well-ordered model

4.1. NOTATION. (1) Let $N_{\lambda,\kappa} = (\lambda, \mathrm{or}, <, \mathrm{or}_1, \mathrm{pa}, \mathrm{pr}_1, \mathrm{pr}_2, 0, S, +, \times)$ where (for cardinals $\lambda, \kappa$) $\mathrm{or} = \lambda$, $\mathrm{or}_1 = \kappa$, $<$ is the well ordering of the ordinals, pa is a Gödel pairing function, $\mathrm{pr}_1$, $\mathrm{pr}_2$ its projections (so that

[†] I.e. we replace the combinatorics there by 2.5 here.

$\mathrm{pa}(\mathrm{pr}_1(\alpha), \mathrm{pr}_2(\alpha)) = \alpha$, and $\mathrm{pr}_l(\mathrm{pa}(\alpha_1, \alpha_2)) = \alpha_l$, 0 is zero, $S$ the successor function, $+$ ordinal addition, and $\times$ ordinal multiplication.

Let $L = \{<, \mathrm{pa}, \mathrm{pr}_1, \mathrm{pr}_2, S, +, \times\}$ and denote

$$\psi_{\mathrm{or}} = \psi^L_{\mathrm{or}}, \qquad \psi_{\mathrm{or}_1} = \psi^L_{\mathrm{or}_1}, \qquad \psi_< = \psi^L_< \qquad \text{etc.}$$

(2) Let $\varphi'_0(u, X, W, W^+, D, D^*)$ say that in $D^*$:

(i) $D \subseteq D^*$, $D$ dense in $D^*$, $X \cap u \subseteq D$ is a dense subset of $D \cap u$, and, for every strongly $(D, X, X)$-perfect set $P$, for some strongly $(D, X, X)$-perfect $P_1 \subset P$, $P_1 \setminus D$ is disjoint to $W^+$,

(ii) for every dense disjoint $E_1$, $E_2 \subseteq X$ and $v \subseteq u$ there is a strongly $(D, E_1, E_2)$-perfect $P \subseteq v$, $P \cap (D^* \setminus D)$ is (non-empty and) $\subseteq W$,

but

(iii) if $P$ is strongly $(D, D, D)$-perfect, $E_1 \subseteq P \cap D \setminus X$, $E_2 \subseteq P \cap D \cap X$, $E_l$ dense in $P$ and $P \cap (D^* \setminus D)$ is dense in $P$ *then* for some $(D, E_1, E_2)$-perfect $P_1 \subseteq P$ we have: $P_1 \cap D^* \setminus D$ is disjoint to $W$ (and necessarily dense in $P_1$). (We can omit $W^+$.)

(3) $\varphi_0(u, X, W, W^+, D, D^*)$ says: for every $u' \subseteq u$ for some $u'' \subseteq u'$, $\varphi'_0(u'', X, W, D, D^*)$.

4.2. Definition. We define a formula $\psi^* = \psi^*(\bar{W}, \bar{D})$ which is the conjunction of sentences saying the properties listed below:

(0) $\bar{D} = \langle D, D^d, D^* \rangle$, $D \subseteq D^d \subseteq D^*$, $D$ and $D^d$ are dense subsets of $D^*$, $W_l \subseteq D^*$, all formulas below (from 4.1 are made to) depend on the $(X_l \cap u)/\equiv$ only and are hereditarily in $u$ and are relativized to $D^*$. (*Note*: $D$, $D^d$, $D^*$ correspond to $D$, $\bigcup_{i<\kappa} D_i$, $Y$ in 2.5, but see 3.1A(2).)

(A)(a) $\psi_{\mathrm{or}}(u, X, \bar{W}, \bar{D})$ implies $X \cap u$ is a dense subset of $D \cap u$, $u$ open non-empty, and: $\psi_{\mathrm{or}_1}(u, X, \bar{W}, \bar{D})$ *iff* $\psi_{\mathrm{or}}(u, X, \bar{W}, \bar{D}) \wedge X \subseteq^* D^d$.

(b) Equality:

$$\psi_{\mathrm{or}}(u, X_1, \bar{W}, \bar{D}) \wedge \psi_{\mathrm{or}}(u, X_2, \bar{W}, \bar{D}) \Rightarrow$$
$$u \equiv \mathrm{val}_v[(X_1 \cap v = X_2 \cap v) \vee X_1 \cap X_2 \cap v = \varnothing].$$

(c) Linear ordering:

(i) $\bigwedge_{l=1}^2 \psi_{\mathrm{or}}(u, X_l, \bar{W}, \bar{D}) \Rightarrow$
$$u \subseteq^* \mathrm{val}_v[\psi_<(v, X_1, X_2, \bar{W}, \bar{D}) \vee X_1 \cap u \equiv X_2 \cap u \vee \psi_<(v, X_2, X_1, \bar{W}, \bar{D})],$$

(ii) $\varnothing \equiv \mathrm{val}_u(\psi_<(u, X_1, X_1, \bar{W}, \bar{D}))$,

(iii) $\mathrm{val}_u \psi(u, X_1, X_3, \bar{W}, \bar{D}) \subseteq^*$
$$\mathrm{val}_u \psi_<(u, X_1, X_2, \bar{W}, \bar{D}) \cap \mathrm{val}_u \psi_<(u, X_2, X_3, \bar{W}, \bar{D}),$$

(iv) $\psi_<(u, X_1, X_2, \bar{W}, \bar{D})$ implies $X_1 \cap X_2 \cap u \equiv \varnothing$.

(d) All reasonable information on $0, S, +, \times, \mathrm{pa}, \mathrm{pr}_1, \mathrm{pr}_2$ (including their inductive definitions).

(e) $\psi_{\mathrm{or}_1}$ is an initial segment.

(f) If $E_1$, $E_2 \subseteq D$, $P$ is a strongly $(D, E_1, E_2)$-perfect set then there is a hereditarily strongly $(D, E_1, E_2)$-perfect set $P_1 \subseteq P$.

(B)(a) *Coding*:

if $\psi_{\mathrm{or}}(u, X, \bar{W}, \bar{D})$ *then* for some $v \subseteq u$ there are $W_{X,v}$, $W^+_{X,v} \subseteq v \cap D^*$ such that $\vDash \varphi'_0(v, X, W_{X,v}, W^+_{X,v}, \bar{D})$.

(b) *Well ordering*:

for the $\theta$'s listed below: for any $u$ and $\bar{Z}$, *if* $(\exists X)[\psi_{\mathrm{or}}(u, X, \bar{W}, \bar{D}) \wedge \theta(u, X, \bar{Z})]$ *then* for some $X$ and $u' \subseteq u$: $\psi_{\mathrm{or}}(u', X, \bar{W}, \bar{D}) \wedge \theta(u', X, \bar{Z})$ and:
$\psi_{\mathrm{or}}(u', Y', \bar{W}, \bar{D}) \wedge \theta(u', Y', \bar{Z})$ implies
$u \subseteq^* \mathrm{val}_v[Y' \cap v = X \cap v$ or $\psi_<(v, X, Y', \bar{W}, \bar{D})]$.

The list of $\theta$'s is:

(i) $\theta_1(u, X, \bar{Z}) \stackrel{\mathrm{def}}{=} \psi_{\mathrm{or}}(u, X, \bar{W}, \bar{D}) \wedge (X \cap u \subseteq^* Z \cap u]$ so $\bar{Z} = \bar{W} \,\hat{}\, \bar{D} \,\hat{}\, \langle Z \rangle$,

(ii) $\theta_2(u, X, \bar{Z}) \stackrel{\mathrm{def}}{=} \psi_{\mathrm{or}}(u, X, \bar{W}, \bar{D}) \wedge Z \subseteq D^* \setminus D$
$\wedge (\forall u' \subseteq u)(\forall E)$
[*if* $E \subseteq u' \cap X$ is dense in $u'$ *then* there is a strongly $(D, E, E)$-perfect $P$, $D^* \cap (P \setminus D) \subseteq Z]$,

(iii) $\theta_3(u, X, \bar{Z}) = \varphi'_0(u, X, W, W^+, \bar{D}') \wedge X \subseteq X^*$.

(c) If $\varphi_0(u, X, \bar{W}, \bar{D})$ then, for every $u^1 \subseteq u$, for some $u^2 \subseteq u^1$ there is $Z \subseteq D^* \setminus D$ such that:

(i) for every $E \subseteq u^2 \cap X$ dense in $u^2$ there is a strongly $(D, E, E)$-perfect $P$, $D^* \cap (P \setminus D) \subseteq Z$,

(ii) for every $(D, (D \setminus X) \cap u^2, (D \setminus X) \cap u^2)$-perfect $P$, there is a strongly $(D, D \setminus X, D \setminus X)$-perfect $P' \subseteq P$ such that:
$D^* \cap (P' \setminus D) \cap Z = \varnothing$.

(d) *Distributivity*:

if $\psi_{\mathrm{or}_1}(u_1, X_1, \bar{W}, \bar{D})$, then there is $Y_1 \subseteq D \cap u_1$ such that:

(i) assume $v \subseteq u_1$, $\psi_{\mathrm{or}}(v, X, \bar{W}, \bar{D})$; we have:
$\psi_<(v, X, X_1, \bar{W}, \bar{D})$ iff $X_1 \cap v \subseteq^* Y_1$,

(ii) if $Y \subseteq Y_1$ and
$(\forall X)[\psi_<(v, X, X_1, \bar{W}, \bar{D}) \wedge v \subseteq^* u_1 \Rightarrow Y \cap X \cap v$ is nowhere dense]
*then* $Y$ is nowhere dense,

(e) if $\psi_{\mathrm{or}}(u_1, X_1, \bar{W}, \bar{D}) \wedge \neg\, \psi_{\mathrm{or}_1}(u_1, X_1, \bar{W}, \bar{D})$ then for any $Y_1$, (i) or (ii) of (d)(b) fails for $u_1$, $X_1$.

4.3. FACT. If $N = N_{\lambda,\kappa}$ (see 4.1) and $\lambda$, $\kappa$, $S_1$, $S_2$, $T$, $D$, $Y$ as in 2.5,

$D^* \stackrel{\text{def}}{=} Y$, the set $\{\delta < \lambda^+ : S_1 \cap \delta$ is not stationary, $\mathrm{cf}(\delta) = \kappa\}$ is stationary, $Z$ a subspace of $\mathrm{Top}_{\mathrm{lx}}(T)$, $D^* \subseteq Z$, and $M = M_Z$, *then* for some $\bar{W}$,

$$M \models \psi^*[\bar{W}, \bar{D}].$$

PROOF. Immediate: 2.5 is tailored for Definition 4.2, and note that $\kappa$-distributivity by 2.4(3).

4.4. MAIN INTERPRETATION LEMMA. *Suppose*[†] $M \models \psi^*[\bar{W}, \bar{D}]$ *and*

$(*)_1$ *$M$ (or at least some $D'$, $D \subseteq D' \subseteq D^*$, $D' \setminus D$ dense) is a first countable (Hausdorff) space and $D$ is the union of $\aleph_0$ scattered sets*

*or*

$(*)_2$ *$M$ is $\aleph_1$-distributive*

*or*

$(*)_3$ *the topology on $D^*$ is induced by a dense linear order and is, on $D$, first countable.*

Then *for every $u_0$ (open subset of $D^*$, as usual) for some $u \subseteq u_0$ the following holds.*

*There are $\alpha$, and $D_i$ ($i < \alpha$) and $\gamma(*)$ such that*

(a) $\models \psi_{\mathrm{or}}[u, D_i, \bar{W}, \bar{D}]$,

(b) *there are no $v \subseteq u$ and $D'$ such that*

$$\psi_{\mathrm{or}}[v, D', \bar{W}, \bar{D}],$$

$$\psi_{<}[v, D', D_i, \bar{W}, \bar{D}],$$

$$\psi_{<}[v, D_j, D', \bar{W}, \bar{D}] \qquad \text{for } j < i,$$

(c) $D_i \subseteq D^d$ *iff* $i < \gamma(*)$ *iff* $D_i \cap D^d \neq \varnothing$,

(d) *if* $\psi_{\mathrm{or}}(u, D', \bar{W}, \bar{D})$ *then*
$u \subseteq^* \mathrm{val}_v(\bigvee_i D' \cap v = D_i \cap v)$,

(e) for $i < \gamma(*)$, *there is $Y_i$ such that*:

(i) $D_j \subseteq^* Y_i$ *for* $j < i$,

(ii) $D_j \cap Y_i \equiv \varnothing$ *for* $j \geqq i$,

(iii) $(\forall X \subseteq Y_i)[\bigwedge_{j<i} D_j \cap X \equiv \varnothing \Rightarrow X \equiv \varnothing]$ *iff* $i < \gamma(*)$.

(f) $u \Vdash_{Q(M)}$ *"there are no new bounded subsets of $\gamma(*)$" at least if $(*)_3$; really* $u \Vdash_{Q(M)}$ "$\underset{\sim}{\kappa}(M) = \gamma(*)$" (see Definition 5.2(2) on $Q(M)$, $\underset{\sim}{\kappa}(M)$).

[†] $M$ the monadic topology of a topological space which we denote by $M$, too; see Definition 1.2.

4.4A. REMARK. Seemingly in $(*)_3$ [$x \in D \Rightarrow x$ has confinality $\aleph_0$ from at least one side] suffice.

PROOF. We shall first try to define $D_i$ $(i < \alpha)$ satisfying (a), (b), (c), (d).

So we let first $u = u_0$, and start to choose $D_i \subseteq D^d$ satisfying (a), (b) and (c). So for some $\beta$, $\langle D_i : i < \beta \rangle$ is defined, but we cannot define $D_\beta$. If for some $u^* \subseteq u$,

$(*)$ for every $D'$:

$$\psi_{\mathrm{or}}(u^*, D', \bar{W}, \bar{D}) \Rightarrow u^* \subseteq \mathrm{val}_v\left( \bigvee_{i<\beta} D' \cap v = D_i \cap v \right)$$

*then* we could have chosen $u = u^*$, so we succeed (it is easy to choose $\gamma(*)$).

Next suppose there is no such $u^*$, but for every $v_1 \subseteq u$ there are $v_2 \subseteq v_1$ and $D'$ such that:

$$\psi_{\mathrm{or}}(v_2, D', \bar{W}, \bar{D}),$$

$$v_2 \subseteq^* \mathrm{val}_w \psi_<(w, D_i, D', \bar{W}, \bar{D}),$$

and for every $D''$:

$$\left[ \bigwedge_{i<\beta} v_2 \subseteq^* \mathrm{val}_w \psi_<(w, D_i, D'', \bar{W}, \bar{D}) \right]$$
$$\Rightarrow v \subseteq^* \mathrm{val}_w \psi_<(D'' \cap w = D' \cap w \vee \psi_<(w, D', D'', \bar{W}, \bar{D}))$$

*then* we can contradict the choice of $\beta$.

So for some $v_1 \subseteq u$ for no $v \subseteq v_1$ the statement above holds.

We shall get a contradiction to the well ordering. Quite easily, we can build $X_n$,

$$M \models \psi_{\mathrm{or}_1}[v_1, X_n, \bar{W}, \bar{D}],$$

$$M \models \psi_<[v_1, D_i, X_n, \bar{W}, \bar{D}] \qquad \text{for } i < \beta,$$

$$M \models \psi_<[v_1, X_n, X_{n+1}, \bar{W}, \bar{D}].$$

We want to get a contradiction to the well-ordering requirement ((B)(b) of 4.2).

The proof of this splits into three cases, according to which of the alternative assumptions of 4.5 holds.

*Case* 1. $(*)_1$ holds.

Remember that for any $v \subseteq v_1$ and $n$ for some $v' \subseteq v$ and $W_{X_n} \subseteq (D^* - D) \cap v'$:

$$M \models \varphi_0'[v', X_n, W_{X_n}, W_{X_n}^+, \bar{D}] \qquad \text{(see (B)(a) of 4.2).}$$

Let $\{(w_\alpha^n, W_{X_n,\alpha}, W_{X_n,\alpha}^+) : \alpha < \alpha_n\}$ be such that $\{w_\alpha^n : \alpha < \alpha_n\}$ is a maximal family of pairwise disjoint (regular open non-empty) subsets of $\omega_1$, $W^{\alpha}_{X_n} \subseteq \omega_\alpha$, $M \models \varphi'_0[\omega^n_\alpha, X_n, W_{X_n,\alpha}, W^+_{X_n,\alpha}, \bar{D}]$ (see 4.1(3), 4.1(2)). Let $W_{X_n} = \bigcup_{\alpha<\alpha_n} W_{X_n,\alpha}$ and $W^+_{X_n} = \bigcup_{\alpha<\alpha_n} W^+_{X_n,\alpha}$. Let $W^* = \bigcup_n W_{X_n}$ and $W^+ \stackrel{\text{def}}{=} \bigcup_{n<\omega} W_{X_n}$. Clearly $\models \varphi_0(\omega_1, X_n, W^*, W^+, \bar{D}]$.

[Why? Checking Definition 4.1(2), (i) is proved like (iii) below, (ii) holds easily as $W_{X_n} \subseteq W^*$; as for (iii): if $P$, $E_1$, $E_2$ are as there, by 2.3(6), (7) w.l.o.g. every perfect $P' \subseteq P$ satisfies $P' \setminus D$ is dense in $P'$; we use repeatedly 4.1(2)(iii) for $\varphi'_0(\omega^n_\alpha, X_n, W^\alpha_{X_n}, \bar{D})$ and first countability of $D$, to find $P' \subseteq P$ a $(D, E_1, E_2)$-perfect set such that $(P' - D) \cap W'_{X_m} = \emptyset$ for each $m$, and it is as desired.]

Now there is a $Y' \subseteq \bigcup_{n<\omega} X_n \cap \omega \subseteq D \cap \omega$, (dense) such that

$$\models \varphi'_0(\omega_1, Y', W^*, W^+, D, D^*) \wedge \varphi_{\mathrm{or}}(\omega_1, Y', \bar{W}, \bar{D})$$

and

$$[\varphi_0(\omega_1, Z, W^*, \bar{D}) \wedge \psi_{\mathrm{or}}(\omega_1, Z, \bar{W}, \bar{D})]$$
$$\Rightarrow \omega_1 \subseteq \mathrm{val}_\omega[Z \cap \omega = Y \cap \omega \text{ or } \psi_<(\omega, Y', Z, \bar{W}, \bar{D})]$$

(see 4.2(B)(b), i.e. $\theta_3 = \varphi'_0$ & $X \subseteq \bigcup_n X_n$). *Note*: $Y' \cap X_n \cap \omega_1 \equiv \emptyset$ (by (A)(c)(iv) of Definition 4.1).

We can now define $E_1$, $E_2$ such that: $E_1$, $E_2$ are dense in $\bigcup_{n<\omega} X_n \cap \omega \subseteq D^*$, disjoint, $E_1 \cup E_2 \subseteq Y'$ but for each $n$ $(E_1 \cup E_2) \cap (\bigcup_{l<n} X_l)$ is scattered (use first countability and "$D$ is the union of $\aleph_0$ scattered sets" from $(*)_1$).

Let $P$ be a strongly $(D, E_1, E_2)$-perfect subset of $D^*$ such that $P \cap D^* \setminus D \subseteq W^*$ (exists by 4.1(2)(ii)).

Now by the first countability by successive approximations we can find $P_1 \subseteq P$, $P_1 \cap E_l \subseteq P_1$ is dense in it, $(P_1 \setminus D) \cap W_{X_l} = \emptyset$ for each $l$.

*Case* 2. $\aleph_1$-distributivity.
Easy.

*Case* 3. $(*)_3$ holds.
Define $\{(\omega^n_\alpha : {}^\alpha_{X_n}) : \alpha < \alpha_n\}$, $W_{X_n}$, $W^*$ as in Case 1.
W.l.o.g. each $\omega^n_\alpha$ is an interval and
$(*)$ $\forall \beta < \alpha_{n+1} \exists \gamma < \alpha_n[\omega^{n+1}_\beta \subseteq \omega^n_\gamma]$.
If for some $\langle \beta_n : n < \omega \rangle$, $\beta_n < \alpha_n$, $|\bigcap_n \omega^n_{\beta_n}| > 1$, then we get a contradiction as in Case 2.

Otherwise choose, by induction on $n$, distinct $a^n_\alpha$, $b^n_\alpha \in \omega^n_\alpha$ which are not in $\{a^m_\beta, b^m_\beta : m < n, \beta < \alpha_m\}$ (really we should consider only finitely many such elements by $(*)_3$). Let

$$E_1 = \{a^n_\alpha : n < \omega, \alpha < \alpha_n\} \quad \text{and} \quad E_2 = \{b^n_\alpha : n < \omega, \alpha < \alpha_n\}.$$

Let $P$ be a strongly $(D, E_1, E_2)$-perfect subset of $D^*$ such that $P \setminus D \subseteq W^*$ and finish as in Case 1.

## §5. Conclusions: Monadic logic is hard

5.1. FACT. In the class of monadic topologies we can define the following classes (each by one sentence):

(a) Hausdorff, regular, normal.

(b) $\mathrm{TOP}_{\mathrm{lin}}$: the class of topologies defined by a complete dense linear order (and reconstruct the order up to inversion).

(c) $\mathrm{TOP}^{\omega_1}_{\mathrm{lin}}$: the class of topologies in $\mathrm{TOP}_{\mathrm{lin}}$ such that the linear order densely contains monotonic $\omega_1$-chains.

(d) $\mathrm{TOP}^{\omega}_{\mathrm{lin}}$: the class of topologies in $\mathrm{TOP}_{\mathrm{lin}}$ such that the linear order has a dense set each member of which has cofinality $\aleph_0$ (from both sides).

5.2. DEFINITION. (1) $Q(M)$ is the forcing notion of open subsets of a topological space $M$ with inverse inclusion.

(2) $\underset{\sim}{\kappa}(M)$ is the $Q(M)$-name expressing the distributivity of $Q(M)$. Equivalently, $\underset{\sim}{\kappa}(M)$ is the first $\kappa$ such that $[\mathscr{P}(\kappa)]^{V^{Q(M)}} \neq [\mathscr{P}(\kappa)]^V$.

5.3. THEOREM. (1) *We have a recursive function $\theta \mapsto \theta^{[l]}$ for $l = 1, 2, 3$ from the set sentences of monadic topologies to the set of sentences in monadic logic such that for*

$$M \in K_{fl} = \{M : \vDash (\exists \bar{D}, \bar{W})\psi^*[\bar{D}, \bar{W}] \textit{ and } M \textit{ is first countable}$$

$$\textit{and } M \textit{ is induced by a linear order}\}.$$

$M \vDash \theta^{[1]}$ iff $\Vdash_{Q(M)}$ "$\underset{\sim}{\kappa}(M) \vDash \theta$";

(2) *if in $\theta$ we quantify only on relations of power smaller than that of the model's power*, then *for each regular $\mu$: there is $M \in K_{fl}$, $\underset{\sim}{\kappa}(M) = \mu$, $M \vDash \theta^{[2]}$* iff *$\mu \vDash \theta$;*

(3) *$\theta$ has a model* iff *$\theta^{[2]}$ has a model in $K_{fl}$, but if they have models*

$$\mathrm{Min}\{2^{\|M\|} : M \vDash \theta^{[3]}\} \geqq \mathrm{Min}\{\lambda : \lambda \vDash \theta\}.$$

PROOF. Straightforward by 2.4, 4.3, 4.4 with $(*)_3$ (or see the proofs in Gurevich–Shelah [GuSh151] or [Sh205, §1]). Remember 1.1A(2).

Note that we should be able to characterize a class of $(M, \bar{W}, \bar{D})$ such that, on the one hand, 4.4 apply to each and, on the other hand, it contains enough $M$'s (e.g. from 4.3, i.e. 2.5).

5.3A. THEOREM. *If $K^*$ is a class of topologies, which include $M_{\mathrm{TOP}(L)}$ where $L$ is the completion of the linear order $(T, <_{lx})$, $T$ from* 2.2, TOP$(L)$ *the topology on $L$ with based open intervals*, then *in* 5.3 *we can vary $M$ on all members of $K^*$.*

PROOF. By 5.1(b),(d).

5.3B. THEOREM. *In* 5.3, 5.3A *we can let $M$ vary over linear orders* (*i.e.*, $\theta$ *vary on the sentence in monadic logic for linear orders*).

(Here we do not need completeness.)

PROOF. Immediate from the proofs of 2.5, 4.3, 4.4, 5.3.

5.4. THEOREM. *Let*

$$K = \{\lambda : \text{the consequences of } 2.6 \text{ hold (with } \lambda \text{ here standing for } \lambda^+ \text{ there) for } \kappa = \aleph_0, \text{ e.g. } (\exists \mu)(\lambda = (\mu^{\aleph_0})^+)\}.$$

*For $l = 1, 3$, there are recursive maps $\theta \mapsto \theta^{[l]}$, such that:*

(0) *For every sentence $\theta$ in pure second order logic, $\theta^{[l]}$ is in monadic topology.*

(1) *For a metrizable topological space $X$ with no isolated points* $\Vdash_{Q(M_X)}$ "$\underset{\sim}{\kappa}(M) = \aleph_0$".

(2) *For a monadic topology with no isolated points* $M \in K_{cm} = \{M_X : X$ *a completely metrizable space and locally the density of $X$ is in* $\kappa\}$:
$M \models \theta^{[1]}$ *iff* $\Vdash_{Q(M)}$ "$\underset{\sim}{\kappa}(M) \models \theta$".

(3) *If $\lambda \in K$,* $\Vdash_{\mathrm{Levy}(\aleph_0, \lambda)}$ "$\lambda \models \theta$" *iff for some completely metrizable space $M$, $M \models \theta^{[3]}$ where density $(M \restriction u) = \lambda$ for every $u$*

(4) $\forall M \models \theta^{[3]} \Rightarrow (\exists \lambda)[\Vdash_{\mathrm{Levy}(\aleph_0, \lambda)} \lambda \models \theta \wedge \lambda \leqq 2^{\|M\|} \wedge \lambda \geqq \mathrm{Min}\{\text{density of } M \restriction u : u\}]$.

PROOF. We use §2(2.6), §3, §4 for $\kappa = \aleph_0$.

We lose our ability to say "the space is induced by a linear order (and is first countable)", but first countability and $(*)_1$ of 4.5 are given.

Note that we use:

(α) $\Vdash_{\mathrm{Levy}(\aleph_0,\lambda)}$ "$\lambda \models \theta$" *iff* $\Vdash_{\mathrm{Levy}(\aleph_0,\lambda)}$ "$\aleph_0 \models \theta$" (so we can work as in [Sh205, §1]),

(β) if $M$ is a dense completely metrizable space, then cellularity is equal to density.

5.5. REMARK. (1) The interpretations are sematic, but not strictly in the classical sense; see Baldwin, Shelah [BlSh156] and Gurevich–Shelah [GuSh168].

(2) We may interpret, say in the topologies like $\lambda^{\omega}$, second-order logic on the cardinal $\lambda^{\aleph_0}$. For $\lambda = \aleph_0$ this is done in detail in Part A; generally it probably works at least for $\lambda$ strong limit of cofinality $\aleph_0$, but I have not the time to try.

(3) We can also deal with restricted classes of linear orders.

5.6. REMARK. Generally for any class $K$ of topologies, we can interpret $\{\mathrm{Th}^{Q(M)}_{\mathrm{ind}}(\underset{\sim}{\kappa}) : M \in K', (M, \bar{D}^*, \bar{W}) \models \psi\}$ where $K' = \{M \in K$: the analysis of §4 apply$\}$. So then our class has to contain complete linear order.

5.7. REMARK. The "no isolated point" clause is added just to clarify. But this is serious if our interest is in topological spaces ${}^{\omega \geqq}\lambda$ with the topology

$$\left\{ u\!\!\!\!\!\,u: u\!\!\!\!\!\,u \subseteq {}^{\omega \geqq}\lambda \quad \text{and} \quad \eta \in u\!\!\!\!\!\,u \cap {}^{\omega}\lambda \Rightarrow \bigvee_n \{\eta \restriction m : n < m < \omega\} \subseteq u\!\!\!\!\!\,u \right\}.$$

We can handle them similarly.

## §6. Consequences related to [BlSh156]

See Baldwin, Shelah [BlSh156] and [Sh284C].

Let $\mathcal{T}$ denote a first-order theory.

6.1. THEOREM. (1) *If some monadic expansion of a model of $\mathcal{T}$ is unstable*, then *the Lowenheim number of* ($\mathcal{T}$, Mon) *is at least that of second-order logic*.

(2) *Suppose $\mathcal{T}$ is not superstable*, ($\mathcal{T}_\infty$, 2nd)$\not\leqq$($\mathcal{T}$, Mon), $\mathcal{T}$ *had* NDOP *and a finite language. Then in the monadic theory of the class of models of $\mathcal{T}$ we can interpret the theory of the family of topological spaces which are closed subsets of some ${}^{\omega}\lambda$ (hence complete metric spaces).*

6.1A. REMARK. We can use different coding: essentially we ask for perfect subtrees (closed downward) such that the splitting points are only in $E_1$, $E_2$ — and in each densely. It is not clear whether this has any extra application.

## Appendix: The black box

The following theorem is a reformulation of [Sh300, second version], III, 6.12 (and 6.12); generally on black boxes and references there.

We will use the case $|L| = \kappa$, $\sigma = \aleph_0$, $\mu = \kappa^+$.

A.1. THEOREM. *Suppose* $\lambda^\kappa = \lambda$, $S \subseteq \{\delta < \lambda^+ : \operatorname{cf}\delta = \aleph_0\}$ *is stationary,* $\lambda^+ \subseteq A$, $|A| = \lambda^+$, $f: A \to \lambda^+$, *$L$ a vocabulary with* $\leqq \lambda$ *predicate and function symbols, each with* $< \sigma$ *places,* $\kappa^{<\sigma} = \kappa$. *Then we can find* $\langle \langle N_i^\alpha : i \leqq \omega \rangle : \alpha < \lambda^+ \rangle$, *and functions* $\zeta$, $h_{\alpha,\beta}(\alpha, \beta < \lambda^+, \lambda(\alpha) = \lambda(\beta))$ *such that*:

(A)(a) *$N_i^\alpha$ is a model of cardinality* $\leqq \kappa$, *universe* $\subseteq A$, *and vocabulary* $L_i^\alpha \subseteq L$ *of cardinality* $\leqq \kappa$; *$N_i^\alpha$ closed under $f$ and* $f^{-1}$,

(b) *for* $i < j \leqq \omega$, $L_i^\alpha \subseteq L_1^\alpha$, $N_i^\alpha \subseteq N_j^\alpha$ (*i.e.* $N_i^\alpha \subseteq N_j^\alpha \restriction L_i^\alpha$) *and if* $j < \omega$, $N_i^\alpha \prec N_j^\alpha$ (*so* $\sigma = \aleph_0$, $j = \omega$ *is O.K.*),

(c) $N_\omega^\alpha = \bigcup_{n<\omega} N_n^\alpha$,

(d) *$\zeta$ is a function from $\lambda^+$ to* $S(\subseteq \lambda^+)$, *monotonically increasing* (*not strictly*), $\zeta(\alpha) = \sup(N_\omega^\alpha \cap \lambda^+)$.

(B)(a) Isomorphism: *If* $\zeta(\alpha) = \zeta(\beta)$ then *$h_{\alpha,\beta}$ is an isomorphism from $N_\omega^\beta$ onto $N_\omega^\alpha$, which maps $N_n^\beta$ onto $N_n^\alpha$ (for $n < \omega$), commute with $f, f^{-1}$, preserve the order of the ordinals and maps* $N_\omega^\alpha \cap \lambda$, $N_\omega^\alpha \cap \lambda^+$ *onto* $N_\omega^\beta \cap \lambda$, $N_\omega^\beta \cap \lambda^+$.

(b) Commutativity: *If* $\zeta(\alpha) = \zeta(\beta) = \zeta(\gamma)$ *then* $h_{\alpha,\gamma} = h_{\alpha,\beta} \circ h_{\beta,\gamma}$, $h_{\gamma,\alpha} = h_{\alpha,\gamma}^{-1}$, $h_{\alpha,\alpha} = \mathrm{id}$.

(c) Treeness: *If* $\zeta(\alpha) = \zeta(\beta)$ then $N_\omega^\alpha \cap \lambda = N_\omega^\beta \cap \lambda$, *and* $i \in \lambda^+ \cap N_\omega^\alpha \cap N_\omega^\beta$ *implies* $N_\omega^\alpha \cap i = N_\omega^\beta \cap i$ (*and* $h_{\beta,\alpha} \restriction (N_\omega^\alpha \cap i) = \mathrm{id}$).

(d) *There are* $\langle \eta_\delta : \delta \in S \rangle$ *such that*:
*$\eta_\delta$ is a strictly increasing function from $\omega$ to $\delta$,* $\sup\{\eta_\delta(n) : <\omega\} = \delta$, *and* $\zeta(\alpha) = \delta = \zeta(\beta)$ *implies*:
*for each $n$,* $N_\omega^\alpha \cap \eta_\delta(n) = N_\omega^\beta \cap \eta_\delta(n)$ *and $h_{\alpha,\beta}$ maps, for each $n$,* $N_\omega^\alpha \cap \eta_\delta(n)$ *onto* $N_\omega^\beta \cap \eta_\delta(n)$ *and* $\{\eta_\delta(n) : n < \omega\}$ *is disjoint* $N_\omega^\alpha$.

(C) Density: *In the following game, player II has no winning strategy*:
*The play makes the last $\omega$ move.*
*On the nth move, player I chooses a set $a_n \subseteq A$ of cardinality $\leqq \kappa$, and then player II chooses a model $N_n$,* $a_n \subseteq |N_n|$, *such that* $\langle N_l : l \leqq n \rangle$ *satisfies the relevant parts of* A(a), A(b).
*Player I wins if the play for some $\alpha$,* $\bigwedge_n N_n = N_n^\alpha$.

(D) *For some $\lambda^*$ (not depending on $\lambda$) we can require the following*:

$(*)^1_{\lambda^*}$ *for each $\zeta$, no subset of* $\{N_\omega^\alpha : \zeta(\alpha) = \zeta\}$ *is $\lambda^*$-perfect (of density character $> \lambda^*$) with the natural topology: a neighbourhood of $N_\omega^\alpha$ is* $\{N_\omega^\beta : \zeta(\beta) = \zeta, N_\omega^\alpha \cap i = N_\omega^\beta \cap i\}$ *for some* $i < \zeta$.

A.2. REMARK. This $(*)_{\lambda^*}$ can be done for $\lambda = (2^{\aleph_0})^{+n}$, $\lambda^* = \aleph_0$ (by induction on $n$).

For this, it is enough to prove:

$(*)^2_{\lambda^*}$ there is $A \subseteq {}^{\omega}\lambda$ containing no $\lambda^*$-perfect set, but not disjoint to any $T \subseteq {}^{\omega \geqq}\lambda$ if: $\langle\ \rangle \in T$, $[\eta \notin T \cap {}^{\omega >}\lambda \Rightarrow (\exists^{\lambda}\alpha)\eta\ {}^\wedge \langle\alpha\rangle \in T]$ and $[\bigwedge_{n<\omega} \eta \restriction n \Rightarrow T \Rightarrow \eta \in T]$.

In the case $\lambda = \mu^{\aleph_0}$, $\mu$ strong limit of cofinality $\omega$, $(*)^2_{\lambda^*}$ holds if

$(*)^3$ there is $A \subseteq {}^{\omega}\mu$, $|A| = \mu^{\aleph_0}$, $A$ contains no $\lambda^*$-perfect subset.

Now while this paper was processed, [Sh355], 6.x shows that, for some $\lambda^*$, $(*)^2_{\lambda^*}$ holds (for every $\lambda$).†

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† *Added in proof.* By [Sh400], e.g. for some club $C \subseteq \omega_1$, for $\delta \in C$, $\mu = \beth_\delta$, $(*)_s$ holds. This enables us in 2.6 to find a code for any dense subset (or subsets) of $D$ rather than only for $\langle D_{i/\mathfrak{e}} : i < \lambda \rangle$.

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